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Interation of quasi-rational mapping *

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Abstract The more general case of interation of quasi-rational mappings, including their Julia sets and quasinormal sets, is studied and some results corresponding to complex dynamics are obtained.

Keywords: **quasi-rational mapping, Julia set, quasinormal set.**

1 Definitions and symbols

Suppose that V is a Riemann sphere with unit diameter. There is a one-to-one correspondence between V and the close complex plane $\bar{C}$. This paper will identify them. Let $a, b \in V$. The spherical distance, denoted by $|a, b|$, is the length of the shortest curve joining a, b, i.e. the smaller great circle arc joining a, b on V.

Definition 1. Suppose that $f(z)$ is a continuous complex function in a region $D \subset V$. For every point z_0 of D, if there is a neighborhood $U \subset D$ of z_0, and a positive integer n depending on z_0 such that

$$F(z) = \begin{cases} (f(z))^{\frac{1}{n}}, & f(z_0) = \infty, \\ (f(z) - f(z_0))^{\frac{1}{n}} + f(z_o), & f(z_o) \neq \infty \end{cases}$$

is a univalent K-quasiconformal mapping in U, then f is named K-quasimeromorphic mapping in D, where the extraction of a root may take any branch. If $n \geqslant 2$ in the above formula, then z_0 is named as n valent point or critical point, $m_0 := F(z_0) = f(z_0)$ as critical value, n as valent, and $n-1$ as multiple of critical point or critical value.

Suppose that $f(z)$ is a K-quasimeromorphic mapping in a region $U \subset V$, then it is denoted by $f \in Q_K(V)$. $|U|$ denotes the spherical area of the region U, $|L|$ the length of curve L on sphere V, (U, f), the covering surface generated by f from U to V, and its area is

$$|(U, f)| = \iint_U \frac{|f_z(z)|^2 - |f_{\bar{z}}(z)|^2}{(1 + |f(z)|^2)^2} dx dy, \quad (z = x + iy).$$

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The average covering times of the covering surface (U,f) on V are denoted by

$$S(U,f) = \frac{|(U,f)|}{|V|} = \frac{|(U,f)|}{\pi}.$$

Definition 2. Suppose that $f \in Q_K(C)$. If

$$S(\infty, f) := \lim_{r \to \infty} S(r, f) = \infty,$$

then f is called K-quasitranscendental. If $S(\infty, f) = d(f) < \infty$, then f is called K-quasi-rational, and $d(f)$ is its degree. If f is K-quasi-rational and $f \neq \infty$ always holds, then f is called K-quasipolynomial.

Definition 3. Let $f \in Q_K(V)$ be a K-quasi-rational mapping whose degree $d(f) > K$. $z_0 \in V$ is called a Julia point of f, if for any neighborhood U of z_0, we have

$$\bigcup_n f^n(U) \supset V - \{a, b\},$$

where a, b are two possible exceptional values. The set of all Julia points is called Julia set, denoted by $J(f)$. The set of all exceptional values is called exceptional set, denoted by E. $H(f) := V - J(f)$ is called quasinormal set.

If there is a neighborhood U of z_0 such that the quasi-rational dynamic $\{f^n\}$ ($f^2 = f \circ f$; $f^{n+1} = f \circ f^n$) is normal in U, then z_0 is called a normal point of f. The set of all normal points is called normal set (or Fatou set), denoted by $F(f)$. Obviously $F(f) \subset H(F)$.

Definition 4. Suppose that $f \in Q_K(V)$. If there is a positive number $v < 1$ and a neighborhood U of z_0 such that for any $v \in U$, we have $|f(z), f(z_0)| < v|z, z_0|$, then z_0 is called an attracting point of f. If an attracting point z_0 of f satisfies $f(z_0) = z_0$, then z_0 is called an attracting fixed point of f.

This paper is a continuation of Ref. [1]. In Ref. [1], we proved that $J(f)$ is a non-empty completely invariant perfect set. Please refer to ref. [1 ~ 3] for some other definitions and symbols in this paper.

2 Julia set

Definition 5. Suppose that $f(z) \in Q_K(D)$. We define spherical elasticity rate as

$$f^*(z) = \frac{(|f_z(z)|^2 - |f_{\bar{z}}(z)|^2)(1 + |z|^2)^2}{(1 + |f(z)|^2)^2}.$$

Denote $M = M(D, f) = \sup\{f^*(z);\ z \in D - \Theta\}$, where Θ is the set of all non-differentiable points in D.

Lemma 1. Suppose that $f(z)$ is a K-quasimeromorphic mapping in region $D = \{|z| < R\}$

and continuous on $\bar{D} = \{|z| \leqslant R\}$. If $M = M(D, f) < \infty$, then for any z_0, $\bar{z} \in \bar{D}$, we have

$$|f(z_o), f(\bar{z})| \leqslant \sqrt{MK}\, |z_o, \bar{z}|.$$

Proof. For any $\varepsilon > 0$, connect two points z_0, $\bar{z}$ by a smooth Jordan curve $l(\varepsilon)$ such that its length $|l(\varepsilon)| < (1+\varepsilon)|z_0, \bar{z}|$ and $f(z)$ is absolutely continuous on $l(\varepsilon)$. We take points z_0, z_1, z_2, $\cdots$, $z_n = \bar{z}$ on the curve $l(\varepsilon)$ and denote the section of $l(\varepsilon)$ between z_{j-1}, z_j by $l(j)$, such that

$$|l(j)| = \frac{|l(\varepsilon)|}{n}, \qquad j = 1, 2, \cdots, n.$$

Choose n sufficiently large such that

$$||f(z)| - |f_j|| < \varepsilon, \quad z \in l(j),$$

where $|f_j| = \min\{|f(z)|;\ z \in l(j)\}$, $j = 1, 2, \cdots, n$. Then

$$|f(z_o), f(\bar{z})| \leqslant |f(l(\varepsilon))| = \lim_{n\to\infty}\sum_{j=1}^{n}\frac{|f(z_{j-1}) - f(z_j)|}{\sqrt{1+|f(z_{j-1})|^2}\cdot\sqrt{1+|f(z_j)|^2}}$$

$$= \lim_{n\to\infty}\sum_{j=1}^{n}\frac{\left|\int_{l(j)} f_z \mathrm{d}z + f_{\bar{z}}\mathrm{d}\bar{z}\right|}{\sqrt{1+|f(z_{j-1})|^2}\cdot\sqrt{1+|f(z_j)|^2}}$$

$$\leqslant \lim_{n\to\infty}\sum_{j=1}^{n}\int_{l(j)}\frac{|f_z| + |f_{\bar{z}}|}{1+|f_j|^2}|\mathrm{d}z|.$$

As for any $z \in l(j)$, we have

$$\frac{1+|f(z)|^2}{1+|f_j|^2} \leqslant 1 + 4\varepsilon.$$

Thus

$$|f(z_o), f(\bar{z})| \leqslant (1+4\varepsilon)\lim_{n\to\infty}\sum_{j=1}^{n}\int_{l(j)}\frac{|f_z| + |f_{\bar{z}}|}{1+|f(z)|^2}|\mathrm{d}z|$$

$$= (1+4\varepsilon)\int_{l(\varepsilon)}\frac{\sqrt{|f_z|^2 - |f_{\bar{z}}|^2}}{1+|f|^2}\sqrt{\frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}}\frac{1+|z|^2}{1+|z|^2}|\mathrm{d}z|$$

$$\leqslant (1+4\varepsilon)\sqrt{MK}\int_{l(\varepsilon)}\frac{|\mathrm{d}z|}{1+|z|^2}$$

$$= (1+4\varepsilon)\sqrt{MK}\,|l(\varepsilon)| \leqslant (1+4\varepsilon)(1+\varepsilon)\sqrt{MK}\,|z_o, \bar{z}|.$$

Note ε is an arbitrary positive number. Q.E.D.

Corollary 1. Suppose that the conditions of Lemma 1 hold.

(i) If $MK < 1$, then $z = 0$ is an attracting point of f.

(ii) If $\inf\{f^*(z);\ z \in D - E\} = H > K$, then $z = 0$ is a repelling point of f, i.e. there is $v > 1$ and a neighborhood U of the origin such that $|f(z), f(0)| \geqslant v|z, 0|$ for every $z \in U$.

Proof. (i) The conclusion follows from Lemma 1 immediately. (ii) Take a neighborhood U of the origin such that there is no critical point except possibly $z = 0$. Let

$$\varepsilon = \min\{|f(z), f(0)|;\ z \in \partial U|\}, \quad W_\varepsilon = \{|w, f(0)| < \varepsilon\},$$

$$\delta = \min\{|z|;\ z \in U| \text{ and } f(z) \in \partial W_\varepsilon\} > 0, \quad U_1 = \{|z, 0| < \delta\}.$$

Then for any $z \in U_1 - \{0\}$, we have $f(z) \in W_\varepsilon$. On the Riemann surface (U_1, f) we get a disk W such that its diameter is $f(o)f(z)$. Then there is an inverse mapping $z = g(w)$: $W \to U_1$ of $w = f(z)$ on the disk W, and $z = g(w)$ satisfies the conditions of Lemma 1. Let $v = 1/t = H/K > 1$, $w_0 = f(0)$, and $w = f(z)$. Noting $f^*(z)g^*(w) = 1$, and by Lemma 1, we have

$$|z, 0| = |g(w), g(w_0)| \leqslant t\,|w, w_0| = |f(z), f(0)|/v. \qquad \text{Q. E. D.}$$

Theorem 1. Let $f(z)$ be a K-quasi-rational mapping and its degree $d = d(f) > K$.

(i) The sufficient and necessary condition of $z_0 \in J(f)$ is that for any neighborhood U of z_0, we have

$$\lim_{n\to\infty} \frac{S(U, f^n)}{d^n} > 0.$$

(ii) The sufficient and necessary condition of $z_o \in H(f)$ is that there is a neighborhood U of z_0 such that

$$\lim_{n\to\infty} \frac{S(U, f^n)}{d^n} = 0.$$

(iii) Let D be a close region on the sphere V, then the sufficient and necessary condition of $D \cap J(f) = \varnothing$ is that there is a region U such that

$$\lim_{n\to\infty} \frac{S(U, f^n)}{d^n} = 0.$$

Proof. (i) The conclusion follows from Theorem 4.4 of Ref. [1] and Definition 3.

(ii) By the fundamental inequality of K-quasimeromorphic mappings[2], the condition is clearly sufficient. If the condition is not necessary, then there exist a sufficient small neighborhood U of z_0, with $U \cap J(f) = \varnothing$, a positive number $\varepsilon > 0$ and a sequence $\{n_j\} \to \infty$ such that

$$\frac{S(U, f^{n_j})}{d^{n_j}} > \varepsilon.$$

By the fundamental inequality of K-quasimeromorphic mappings, we have $z_0 \in J(f)$. This is a contradiction.

(iii) The sufficient part holds from Definition 3. If the necessary part is not true, then by unceasingly cutting D, we choose a sequence $\{D_n\}$ of close regions such that $D_1 \supset D_2 \supset \cdots \supset D_m \supset \cdots$

whose diameters tend to zero and

$$\lim_{n\to\infty}\frac{S(U_m,f^n)}{d^n}>0$$

for any m and any region $U_m \supset D_m$. Connecting it with the fundamental inequality of K-quasimeromorphic mappings, we obtain $z_o = \bigcap_{j=1}^{\infty} D_m \in J(f)$. This contradicts the condition. Q.E.D.

Definition 6. Let

$$P(f) = \{u \in J(f);\ \#\{\bigcup_n f^n(u)\} < \infty\},$$

$$R(f) = \{u \in J(f);\ \#\{\bigcup_n f^n(u)\} = \infty \text{ and } \overline{\bigcup_n f^n(u)} = J(f)\},$$

$$B(f) = \{u \in J(f);\ \#\{\bigcup_n f^n(u)\} = \infty \text{ and } \overline{\bigcup_n f^n(u)} \neq J(f)\},$$

where $\#\{\cdot\}$ denotes the number of elements in $\{\cdot\}$. If $z_0 \in R(f)$, z_0 is called a dense orbit point. Obviously, $J(f) = P(f) \cup B(f) \cup R(f)$.

Theorem 2. Let $f(z)$ be a K-quasi-rational mapping and $d = d(f) > K$, then

(i) $P(f)$ is the set of all eventually periodic points in $J(f)$ (an eventually periodic point u means there is a non-negative integer n such that $f^n(u)$ is a periodic point) and countable;

(ii) the dense orbit set $R(f)$ is a non-empty dense subset of $J(f)$;

(iii) $B(f)$ is a non-empty uncountable set.

Proof. (i) By definition 6, we obtain Theorem 2 (i) immediately.

(ii) Let $\{B_j\}$ be a countable open topological base of sphere surface V. For any open set B_j, $\bigcup_n f^{-n}(B_j)$ is an open set too. Note that if $U_j = J(f) \cap (\bigcup_n f^{-n}(B_j))$ is a non-empty set, then U_j is a dense open set on the subspace $J(f)$; thus $R^* = \bigcap_j U_j$ is also a dense open set on the subspace $J(f)$. For any $z \in R^* \subset J(f)$, its forward orbit intersects every non-empty U_j. Hence $R^* \subset R(f)$.

(iii) Choose three open sets U_0, U_1, $U_2 \subset V$ which do not intersect each other such that

$$J_0 = U_0 \cap J(f),\quad J_1 = U_1 \cap J(f),\quad J_2 = U_2 \cap J(f)$$

are all non-empty, and their closures do not intersect each other. Take a positive integer n such that $f^n(J_0) = f^n(J_1) = f^n(J_2) = J(f)$. For any set $E \subset J(f)$, denote

$$F_0(E) = f^{-n}(E) \cap \overline{J_0},\quad F_1(E) = f^{-n}(E) \cap \overline{J_1},\quad F_2(E) = f^{-n}(E) \cap \overline{J_2}.$$

For every triple fraction $b = 0.b_1b_2b_3\cdots \in [0,1]$, ($b_j \in \{0,1,2\}$), set

$$J(b) = \lim_{j \to \infty} F_{b_1} \circ F_{b_2} \circ \cdots \circ F_{b_j}(J(f)),$$

which is non-empty. For an infinite non-cycle binary fraction $a = 0, a_1 a_2 a_3 \cdots \in [0, 1]$, ($a_j \in \{0, 1\}$) and any point $u \in J(a)$, the forward orbit $O^+(u)$ is an infinite set, and $O^+(u) \cap F_2(J(f)) = \emptyset$. Hence $u \in B(f)$. Since the set of all infinite non-cycle binary fractions is uncountable, $B(f)$ is uncountable too. Q.E.D.

Theorem 3. Let $f(z)$ be a K-quasi-rational mapping and $d = d(f) > K$, then f is chaos on $J(f)$, i.e.

(i) the periodic points of f in $J(f)$ are dense on $J(f)$;

(ii) f has topological transmission on $J(f)$, i.e. for two arbitrary non-empty open sets B, $D \subset J(f)$, there is a positive integer n such that $f^n(B) \cap D \neq \emptyset$;

(iii) f has initial sensitivity on $J(f)$, i.e. there is a positive number $\Delta > 0$ for any $x \in J(f)$ and any $\delta > 0$, two points a, $b \in J(f)$ and an integer $n \in N$ always exist such that $|x, a| < \delta$, $|x, b| < \delta$ and $|f^n(a), f^n(b)| > \Delta$.

Proof. By Theorem 5.1 in Ref. [1], we have Theorem 3 (i). By Theorem 5.2 in Ref. [1], we have (ii). Combining Theorem 3 (i) and (ii) with Theorem 2 and noting that there are periodic points and dense orbit points nearby every point $u \in J(f)$, we obtain Theorem 3 (iii). Q.E.D.

Lemma 2. Let $f(z)$ be a K-quasi-rational mapping of $d = d(f) > K$. Suppose that $M = M(D, f) < \infty$, and Ξ is the set of all branching points on the covering surface (V, f), and point $A \in V$ satisfies $|A, \Xi| \geqslant \varepsilon/2$, then for any points a_i, a_j among the inverse image $f^{-1}(A) = \{a_1, a_2, \cdots, a_d\}$ of A, their spherical distance $|a_i, a_j| \geqslant \varepsilon/\sqrt{MK}$.

Proof. Taking two arbitrary different points a, $b \in f^{-1}(A)$, we have $f(a) = f(b) = A$. But $f(a)$ and $f(b)$ do not coincide on the curve (V, f). They belong to different "flakes" of Riemann surface and $f(a)$, $f(b)$ are joined by the image $f(ab)$ of the spherical line segment ab (i.e. the smaller great circle arc joining a, b). Because the curve $f(a, b)$ moves around at least a branching point on (V, f), by Lemma 1, we have

$$|a, b| \sqrt{MK} \geqslant |f(a), f(b)| \geqslant |f(a), \xi| + |\xi, f(b)| \geqslant \varepsilon,$$

where ξ is the nearest branching point to a. Q.E.D.

Lemma 3. Let $f(z)$ be a K-quasi-rational mapping of $d = d(f) > K$ and let $B = B(e, \varepsilon) = \{|e, z| < \varepsilon\}$ be the spherical disk. Suppose that $M = M(D, f) < \infty$, and Ξ is the set of all branching points on the covering surface (V, f). If the spherical distance of the sets Ξ, B (i.e. the infimum of lengths of curves joining Ξ, B) on (V, f) satisfies

$$|\Xi, B| \geqslant \sqrt{MK}\varepsilon,$$

then f is a one-one mapping from B to $f(B)$.

Proof. Otherwise there are two different points a, $b \in B$ satisfying $f(a) = f(b)$ but belonging to different "flakes" of Riemann surface, and $f(a)$, $f(b)$ are joined by the image $f(ab)$ of the spherical line segment ab. Thus the curve $f(ab)$ moves around at least a branching point $\xi \in \Xi$. Suppose that $f(c) = \xi$. By the condition of Lemma 1, the length of $f(ab)$ satisfies

$$|f(a,b)| \geqslant |\xi, f(a)| + |\xi, f(b)| \geqslant 2\sqrt{MK}\varepsilon > \sqrt{MK}\,|a,b|.$$

This contradicts Lemma 1. Q.E.D.

Lemma 4. Suppose that $f(z)$ is a K-quasi-rational mapping of $d = d(f) > K$, and $M = M(D,f) < \infty$, and Ξ is the set of all branching points on the covering surface (V, f). If a set $\Psi = \{a_1, a_2, \cdots, a_q\} \subset V$ $(q \geqslant 8)$ satisfies

(i) $|a, b| \geqslant \varepsilon$ for any two different points a, $b \in \Psi$;

(ii) for any $a \in \Psi$, $|a, \Xi| \geqslant \varepsilon/2$ $(\Psi \cap \Xi = \emptyset)$,

then there exists a subset $\Psi_*^{-1} \subset f^{-1}(\Psi)$ such that

(a) for any two different points a, $b \in \Psi_*^{-1}$, $|a, b| \geqslant \varepsilon/\sqrt{MK}$,

(b) for any $a \in \Psi_*^{-1}$, $|a, \Xi| \geqslant \varepsilon/(2\sqrt{MK})$,

(c) the number of elements in Ψ_*^{-1} satisfies $\# \Psi_*^{-1} \geqslant qd - 2d$.

Proof. By Lemmas 1 and 2, we may get qd different points in $f^{-1}(\Psi)$ satisfying (a). Thus for any $\xi \in \Xi$, there is at most a point of $f^{-1}(\Psi)$ in the spherical disk $\{z;\ |z, \xi| < \varepsilon/(2\sqrt{MK})\}$. As $\# \Xi \leqslant 2(d-1)$, we get

$$\Psi_*^{-1} = f^{-1}(\Psi) - \bigcup_{\xi \in \Xi}\{z;\ |z, \xi| < \varepsilon/(2\sqrt{MK})\}. \qquad \text{Q. E. D.}$$

Theorem 4. Suppose that $f(z)$ is a K-quasi-rational mapping of $d = d(f) > K$, then the box dimension of Julia set is

$$\mathrm{Dim}(J) \geqslant \lim_{\delta \to 0} \frac{\log d/K}{\log \sqrt{f_\delta^* K}},$$

where $f_\delta^* = \sup\{f^*(z);\ z \in J(\delta) - E\}$; $J(\delta) = \{z;\ |z, J| < \delta\}$.

Proof. Take $\Psi = \{a_1, a_2, \cdots, a_8\} \subset J(f)$ such that it satisfies the conditions of Lemma 4. Denote $\Psi_*^{-n-1} = (\Psi_*^{-n})_*^{-1}$. By Lemma 4, we have

$$\# \Psi_*^{-1} \geqslant 8d - 2d > 4d,$$

$$\# \Psi_*^{-2} \geqslant (8d - 2d)d - 2d = 8d^2 - 2d^2 - 2d > 4d^2,$$

$$\# \Psi_*^{-3} \geqslant (8d^2 - 2d^2 - 2d)d - 2d = 8d^3 - 2d^3 - 2d^2 - 2d > 4d^3, \cdots$$

$$\# \Psi_*^{-n} \geqslant (8d^{n-1} - 2d^{n-1} - \cdots - 2d)d - 2d = 6d^n - 2d(d^{n-2} + \cdots + 1)$$

$$= 6d^n - 2d\frac{d^{n-1} - 1}{d - 1} > 4d^n, \cdots$$

and for any a, $b \in \Psi_*^{-n} \subset J(f)$, we have $|a, b| \geqslant \varepsilon(MK)^{-n/2}$. We cover $J(f)$ with a family of spherical disks $B = \{B_j\}$ of radius $r = \varepsilon(MK)^{-n/2}/2$. Since at least $4d^n$ small disks of B are needed to cover Ψ_*^{-n}, and by the definition of box dimension[6], we have

$$\mathrm{Dim}(J) = \lim_{r\to 0}\frac{\log N_r(J)}{-\log r}$$

$$\geqslant \lim_{n\to 0}\frac{\log(4d^n)}{-\log(\varepsilon(MK)^{-n/2}/2)} = \frac{2\log d}{\log MK},$$

where $N_r(J)$ denotes the minimum number of small disks covering $J(f)$ in B. Q.E.D.

3 Quasinormal set

Definition 7. Suppose that f: $V \to V$ is a K-quasi-rational mapping of degree d. For any point $w \in V$, we say that $d_w := d - \#\{f^{-1}(w)\}$ is a deficiency of w and $\sum_{w\in D} d_w$ is the total deficiency of f on region $D \subset V$. Obviously, $d_w \neq 0$ if and only if w is a critical point.

Definition 8. Suppose that f: $V \to V$ is a K-quasi-rational mapping and a set $T \subset V$. If $f(T) = T = f^{-1}(T)$, then T is completely invariant.

Theorem 5. Suppose that f: $V \to V$ is a K-quasi-rational mapping of degree d. If $D \subset V$ is a completely invariant region with Euler characteristic $\rho(D) = \rho$, then the total deficiency of f on the region D equals ρ-ρd.

Proof. Let R_D be the set of all critical values. Then $f(D\text{-}\{f^{-1}(R_D)\})$ is a regular covering surface of d multiple on the region $D - R_D$. Thus we have the following relationship of Euler characteristic:

$$\rho(f(D - \{f^{-1}(R_D))) = d \cdot \rho(D - R_D),$$

$$\sum_{w\in R_D}(d - d_z) + \rho = d \cdot (\#(R_D) + \rho),$$

$$d \cdot \#(R_D) - \sum_{z\in R_D}(d - d_z) = \rho - \rho d.$$

Hence

$$\sum_{z\in R_D} d_z = \rho - \rho d. \qquad \text{Q. E. D.}$$

By Theorem 5, we have

Corollary 2. Suppose that $f: V \to V$ is a K-quasi-rational mapping of degree d,

(i) then the total deficiency of f on the sphere V is $2d-2$.

(ii) if D is a completely invariant simply connected component of the quasinormal set $H(f)$, then the total deficiency of f on D is $d-1$.

Corollary 3. There are at most two completely invariant simply connected components in the quasinormal set $H(f)$.

Theorem 6. Suppose that D is a completely invariant component of the quasinormal set $H(f)$, then the boundary of D is a whole Julia set.

Proof. Since D is completely invariant, its boundary is a non-empty completely invariant close subset of $J(f)$. Therefore, it certainly is a whole Julia set. Q.E.D.

Definition 9. Let u be an attracting fixed point of K-quasi-rational mapping $f(z)$. The region

$$W(u) = \{z \in V;\ f^n(z) \to u\}$$

is called an attracting region. A region $N(u)$ is called a connected attracting normal region, if $N(u)$ is the largest region in all connected regions containing u where the family of functions $\{f^n(z)\}$ is normal. The largest region $A(u)$ in all connected regions that contain u and do not contain Julia point is called a connected attracting region. Obviously $N(u)$ is non-empty and $N(u) \subset A(u)$.

Theorem 7. Suppose that $N(u)$ is a connected attracting normal region of K-quasi-rational mapping of d, then $N(u)$ contains at least a critical value.

Proof. Suppose that there is not any critical value in $N(u)$. Take a disk $U \subset N(u)$ of u such that it satisfies the conditions of Lemma 3. Then we may suppose that the inverse mapping g of f satisfies $g(u) = u$ and $U \subset g(U)$. Let $B_0 := U - \{u\}$, $B_n = g^n(B_0)$, $n = \pm 1, \pm 2, \cdots$. Then $N(u) - \{u\} = \bigcup_n B_n$. On $N(u) - \{u\}$ we define the equivalence relation $\sim$: $x \sim y$ if and only if there is an integer n such that $y = g^n(x)$. Use x_0 to denote the equivalence class containing x. All equivalence classes form a ring surface $T = \{x_0\}$. The natural projections

$$\pi(x) = x_0,\ \pi:\ N(u) - \{u\} \to T$$

form a regular covering surface of T. This contradicts the fact that the universal covering surface of T is the complex plane C. Q.E.D.

Theorem 8. If the number of components of quasinormal set $H(f)$ is finite, then this number equals two at most.

Proof. Suppose that the number of components of quasinormal set $H(f)$ is finite and bigger than two. We take an integer n such that every component of $H(f)$ is completely invariant for the quasi-rational mapping $S = f^n$. By Corollary 3, there is at least a component (denoted by A) not simply connected. Choose a point z in another component D, and a Mobius transformation M moving z to ∞. Then conjugate S by M to get the quasi-rational mapping $F = M \circ s \circ M^{-1}$ such that ∞ belongs to a completely invariant component D' of $H(f)$. Thus the quasinormal set of F contains a multiply connected completely invariant component A' in $V - D'$, and A' is a bounded region in the complex plane C.

Take a loop h in A', such that the region B bounded by h contains at least a Julia point. As A', D' are invariant and $h \subset A'$, for any positive integer m, $F^m(h) \subset A' \subset V - D' \Rightarrow F^m(B) \subset V - D'$ is bounded. This shows F has no Julia point in B. It is a contradiction. Q. E. D.

Corollary 4. Suppose that f has a completely invariant component D in quasinormal set $H(f)$, then

(i) $H(f)$ has at most another completely invariant component;

(ii) the other components of $H(f)$ are all simply connected.

Theorem 9. If u is an attractive fixed point, then $N(u)$ is either simply connected or infinitely connected.

Proof. Take a simply connected closed disk $U \subset A(u)$ such that $u \in U$, $\overline{f(U)} \subset U$ and

$$\partial U \cap (\bigcup_b O^+(b)) = \emptyset,$$

where the sum is taken over all critical points $\{b\}$. Define E_n inductively as follows: $E_0 = U, E_n = f^{-1}(E_{n-1}) \cap A(u)$. Then

$$E_0 \subset E_1 \subset \cdots, \ A(u) = \bigcup_{n=0}^{\infty} E_n.$$

If all E_n are simply connected, then $A(u)$ is simply connected. Otherwise, there is the minimum integer N such that E_N is not simply connected. Hence E_N is an orientable two-dimensional manifold whose boundary ∂E_N has at least two components. Thus $V - E_N$ has at least two components, too. Consider again the branched covers f: $E_{N+k+1} \to E_{N+k}$ $(k = 0, 1, 2, \cdots)$. They are actually covering spaces if $E_{N+k+1} - E_{N+k}$ does not contain any critical points. But the number of boundary curves of E_{N+k} is at least 2^{k+1}. Consequently, $\partial A(u)$ will have infinitely many components and $A(u)$ is infinitely connected. Q.E.D.

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